Stochastic bosonization in arbitrary dimensions

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Abstract. A procedure of bosonization of Fermions in an arbitrary dimension is suggested. It is shown that a quadratic expression in the fermionic fields after rescaling time $t \to t/\lambda^2$ and performing the limit $\lambda \to 0$ (stochastic limit), gives rise to a bosonic operator satisfying the boson canonical commutation relations. This stochastic bosonization of Fermions is considered first for free fields and then for a model with three–linear couplings. The limiting dynamics of the bosonic theory turns out to be described by means of a quantum stochastic differential equations.

In a previous paper [AcLuVo] we have found the following direct relation between quantum theory, in real time, and stochastic processes:

$$\lim_{\lambda \to 0} \lambda A\left(\frac{t}{\lambda^2}\,,\ k\right) = B(t,k) \tag{1.1}$$

where $A(t,k)$ is a free dynamical evolution of a usual (Boson or Fermion) annihilation operator $a(k)$ in a Fock space $\mathcal{F}$, i.e.

$$A(t,k) = e^{itE(k)}a(k) \tag{1.2}$$

k is a momentum variable and $B(t,k)$ is a Boson or Fermion quantum field acting in another Fock space $\mathcal{H}$ and satisfying the canonical commutation relations:

$$[B(t,k), B(t',k')]_{\pm} = 2\pi\delta(t-t')\delta(k-k')\delta(E(k)) \tag{1.2a}$$

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(cf. [AcLuVo] for a precise explanation of this result). Notice that, in (1.1), λ can also be interpreted as the square root of Planck's constant: in this case the limiting relation (1.1) can be interpreted as a new type of semi–classical expansion, describing not the first order approximation to the classical solution, but rather the *fluctuations* around it.

Here λ can be interpreted as the Planck of semiclassical expansion. The relation () is similar to (1.9) and λ plays the role fo $1/\sqrt{N}$.

A (Boson or Fermion) Fock field satisfying commutation relations of the form (1.2a) is called a *quantum white noise* [Ac Fri Lu1] and is a prototype example of quantum stochastic process. In terms of the field operators $\phi(t,\vec{x})$ one can rewrite (1.1) as

$$\lim_{\lambda\to 0} \lambda\phi\left(\frac{t}{\lambda^2}\,,\; x\right) = W(t,x) \tag{1.3}$$

and one can prove that the vacuum correlations of $W(t,x)$ coincide with those of a *classical Brownian motion.*

The scaling $t \mapsto t/\lambda^2$ has its origins in the early attempts by Pauli, Friedrichs and Van Hove to deduce a *master equation* and in the line of research originated from these papers it was used to deduce the effects of the interaction of one or more atoms with a field (or gas) on the dynamics of the atoms.

In a series of papers starting from [AcFrLu] (cf. the introduction of for an historical survey) it has been shown that it is also possible to control the limiting dynamics of the quantum fields themselves under the scaling $t \to t/\lambda^2$ and to prove that the quantum fields converge to quantum Brownian motions and the Heisenberg equation to a quantum Langevin equation. Moreover, as shown in a multiplicity of quantum systems, involving the basic physical interactions, this is a rather universal phenomenon.

This kind of limit and the set of mathematical techniques developed to establish it, was called in [AcLuVo] *the stochastic limit of quantum field theory*. We call it also the $1+3$ asymptotical expansion to distinguish it from analogous scalings generalizing the present one in the natural direction of rescaling other variables beyond (or instead of) time: space, energy particle density,...

The main result of this paper is a generalization of (1.1) to a pair of *Fermi operators* A, A^+ and it can be described by the formula

$$\lim_{\lambda\to 0} \lambda A\left(\frac{t}{\lambda^2}\,,\; k_1\right) A\left(\frac{t}{\lambda^2}\,,\; k_2\right) = B(t,k_1,k_2) \tag{1.4}$$

The remarkable property of the formula (1.4) is that while the $A(t,k)$ are **Fermion** annihilation operators the limit field $B(t,k_1,k_2)$ is a **Boson** annihilation operator which satisfies with its hermitian conjugate, the canonical bosonic commutation relations. For this reasons we call the formula (1.4): *stochastic bosonization.*

The bosonization of Fermions is well known in $1+1$ dimensions, in particular in the Thirring–Luttinger model see, for example [Wh], [SV] and in string theory. The stochastic bosonization (1.4) takes place in the real $1+3$ dimensional space–time and in fact in any dimension. For previous discussion of bosonization in higher dimensions cf. [Lut], [Hal].

In particular in quantum chromodynamics one can think of the Fermi–operators $A(t,k)$ as corresponding to quarks then the bosonic operator $B(t,k_1,k_2)$ can be considered as describing a meson (cf. [AcLuVo qcd] for a preliminary approach to QCD in this spirit).

In the following section we proove the formula (1.4) and then in Section 3 we consider the stochastic bosonization of a model with nontrivial interaction.

§**2.) Boson Fock spaces as stochastic limits of Fermion Fock spaces**

Let $\Gamma_-(\mathcal{H}_1)$ denote the Fermi Fock space on the 1–particle space $\mathcal{H}_1$ and let, for $f\in\mathcal{H}_1$, $A(f)$, $A^+(f)$ denote the creation and annihilation operators on $\Gamma_-(\mathcal{H}_1)$ which satisfy the usual canonical anticommutation relations (CAR):

$$A(f)A^+(g)+A^+(g)A(f)=\langle f,g\rangle \tag{2.1}$$

The main idea of stochastic bosonization is that, in a limit to be specified below, two Fermion operators give rise to a Boson operator. To substantiate the idea let us introduce the operators

$$\mathcal{A}(f,g):=A(g)A(f),\qquad \mathcal{A}^+(f,g):=\big(\mathcal{A}(f,g)\big)^* \tag{2.2}$$

then by the CAR we have that

$$\mathcal{A}(f,g)\mathcal{A}^+(f',g')=A(g)A(f)A^+(f')A^+(g')=$$

$$=<g,g'><f,f'>-<f,f'><g,g'>+\mathcal{A}^+(f',g')\mathcal{A}(f,g)+R(f,g;f',g') \tag{2.3}$$

i.e.

$$[\mathcal{A}(f,g),\mathcal{A}^+(f',g')]=<(f\otimes g),\ (f'\otimes g')>+R(f,g;f',g') \tag{2.4}$$

where we introduce the notations

$$R(f,g;g',f'):=<f,g'>A^+(f')A(g)-<f,f'>A^+(g')A(g)-<g,g'>A^+(f')A(f) \tag{2.5a}$$

$$<(f\otimes g),\ (f'\otimes g')>:=<f,f'><g,g'>-<f,g'><g,f'> \tag{2.5b}$$

Moreover,

$$\mathcal{A}(f,g)\mathcal{A}(f',g')=\mathcal{A}(f',g')\mathcal{A}(f,g) \tag{2.6}$$

In the following the identities (2.4) and (2.6) will be called the *quasi–CCR*. We shall prove that, in the stochastic limit, the *remainder term* R tends to zero so that, in this limit, the *quasi CCR* become *bona fide* CCR.

The first step in the stochastic limit of quantum field theory is to introduce the *collective operators*. In our case we associate to the quadratic Fermion operator (2.2) the collective creation operator defined by:

$$A^+_\lambda(S,T;f_0,f_1):=\lambda\int_{S/\lambda^2}^{T/\lambda^2}e^{i\omega t}A^+(S_tf_0)A^+(S_tf_1)dt \tag{2.7}$$

where ω is a real number (whose physical meaning explained at the end of Section (3.), $S \leq T$, $f_0, f_1 \in \mathcal{K}$ and $S_t : \mathcal{H}_1 \to \mathcal{H}_1$ is the one–particle dynamical evolution whose second quantization gives the free evolution of the Fermi fields, i.e.

$$A(f) \mapsto A(S_t f) \ ; \quad A^+(f) \mapsto A^+(S_t f)$$

The collective annihilation operators are defined as the conjugate of the collective creators.

The introduction of operators such as the right hand side of (2.7) is a standard technique in the stochastic limit of QFT. We refer to [AcAlFriLu] for a survey and a detailed discussion, while here we limit ourselves to state that the choice (2.7) is dictated by the application of first order perturbation theory to the interaction Hamiltonian considered in Section (3) below.

Having introduced the collective operators, the next step of the stochastic approximation is to compute the 2–point function

$$< \Phi, A_\lambda(S,T; f_0, f_1) A^+_\lambda(S', T'; f'_0, f'_1) \Phi >=$$

$$= \lambda^2 \int_{S/\lambda^2}^{T/\lambda^2} dt \int_{S'/\lambda^2}^{T'/\lambda^2} ds e^{i\omega(t-s)} < \Phi, A(S_t f_0) A(S_t f_1) A^+(S_s f'_0) A^+(S_s f'_1) \Phi > \tag{2.8}$$

By the CAR, the scalar product in the right hand side of (2.8) is equal to

$$< S_t f_0, S_s f'_1 >< S_t f_1, S_s f'_0 > - < S_t f_0, S_s f'_0 >< S_t f_1, S_s f'_1 > \tag{2.9}$$

By standard arguments [AcLu1] one proves that the limit, as $\lambda \to 0$, of (2.8) is

$$< \chi_{[S,T]}, \chi_{[S',T']} >_{L^2(\mathbf{R})} \cdot \int_{-\infty}^{\infty} ds e^{i\omega s} \left[< f_0, S_s f'_1 >< f_1, S_s f'_0 > - < f_0, S_s f'_0 >< f_1, S_s f'_1 >\right] \tag{2.10}$$

Let us introduce on the algebraic tensor product $\mathcal{K} \otimes \mathcal{K}$ the pre–scalar product $(\cdot|\cdot)$ defined by:

$$(f_0 \otimes f_1 | f'_0 \otimes f'_1) :=$$

$$= \int_{-\infty}^{\infty} ds e^{i\omega s} \left[< f_0, S_s f'_1 >< f_1, S_s f'_0 > - < f_0, S_s f'_0 >< f_1, S_s f'_1 >\right] \tag{2.11}$$

and denote by $\mathcal{K} \otimes_{\mathrm{FB}} \mathcal{K}$ the Hilbert space obtained by completing $\mathcal{K} \otimes \mathcal{K}$ with this scalar product. That the sesquilinear form (2.11) is positive is not obvious at first sight, but follows from the fact that (2.10) is the limit of (2.8). Now let us consider the correlator

$$< \Phi, \prod_{k=1}^{n} A_\lambda^{\varepsilon(k)}(S_k, T_k; f_{0,k}, f_{1,k}) \Phi > \tag{2.12}$$

where,

$$A^\varepsilon := \begin{cases} A^+, & \text{if } \varepsilon = 1, \\ A, & \text{if } \varepsilon = 0 \end{cases}$$

By the CAR, it is easy to see that

LEMMA Theexpression(2.1) (2.12) is equal to zero if the number of creators is different from the number of annihilators, i.e.

$$\Big|\{k;\ \varepsilon(k)=1\}\Big| \neq \Big|\{k;\ \varepsilon(k)=0\}\Big| \tag{2.13}$$

or if there exsits a $j=1,\cdots,n$ such that the number of creators on the left of j is greater than the number of annihilators with the same property, i.e.

$$\Big|\{k\leq j;\ \varepsilon(k)=1\}\Big| > \Big|\{k\leq j;\ \varepsilon(k)=0\}\Big| \tag{2.14}$$

THEOREM (2.2) The limit, as $\lambda\to 0$, of (2.12) is equal to

$$<\Psi, \prod_{k=1}^{n} a^{\varepsilon(k)}(\chi_{[S_k,T_k]}\otimes f_{0,k}\otimes_{\mathrm{FB}} f_{1,k})\Psi> \tag{2.15}$$

where, a, a^+ and Ψ are (Boson) annihilation, creation operators on the Boson Fock space $\Gamma_+(L^2(\mathbf{R})\otimes\mathcal{K}\otimes_{\mathrm{FB}}\mathcal{K})$ respectively and Ψ is the vacuum vector.

Proof By Lemma (2.1) and the CCR, we need only to prove the statement in the case $n=2N$ and ε such that neither (2.13) nor (2.14) are true. In this case, by the definition of the collective operators, (2.12) is equal to

$$\lambda^{2N}\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1\cdots\int_{S_{2N}/\lambda^2}^{T_{2N}/\lambda^2} dt_{2N} e^{i(-1)^{1-\varepsilon(1)}\omega t_1}\cdots e^{i(-1)^{1-\varepsilon(2N)}\omega t_{2N}}$$

$$<\Phi, A^{\varepsilon(1)}(S_{t_1}f_{\varepsilon(1),1})A^{\varepsilon(1)}(S_{t_1}f_{1-\varepsilon(1),1})\cdots A^{\varepsilon(2N)}(S_{t_{2N}}f_{\varepsilon(2N),2N})A^{\varepsilon(2N)}(S_{t_{2N}}f_{1-\varepsilon(2N),2N})\Phi> \tag{2.16}$$

By the CAR, the scalar product in (2.16) is equal to

$$\sum_{\substack{1\leq j_k,p_k,q_k\leq n,\ \ \epsilon,\epsilon'\in\{0,1\}^{2N}\\ j_k<p_k,j_k<q_k,\varepsilon(j_k)=0,\varepsilon(p_k)=\varepsilon(q_k)=1}} R(\epsilon,\epsilon',\{j_k,p_k,q_k\})$$

$$\prod_{k=1}^{N}<S_{t_{j_k}}f_{0,j_k}, S_{t_{p_k}}f_{\epsilon(p_k),p_k}><S_{t_{j_k}}f_{1,j_k}, S_{t_{q_k}}f_{\epsilon'(q_k),q_k}>=$$

$$=\sum_{1\leq j_k<p_k\leq n,\ \ \epsilon,\in\{0,1\}^{2N}} R(\epsilon,\{j_k,p_k\})$$

$$\prod_{k=1}^{N}<S_{t_{j_k}}f_{0,j_k}, S_{t_{p_k}}f_{\epsilon(p_k),p_k}><S_{t_{j_k}}f_{1,j_k}, S_{t_{p_k}}f_{1-\epsilon(p_k),p_k}>+$$

$$+ \sum_{\substack{1\le j_k,p_k\ne q_k\le n,\ \ \epsilon,\epsilon'\in\{0,1\}^{2N} \\ j_k<p_k,j_k<q_k,\varepsilon(j_k)=0,\varepsilon(p_k)=\varepsilon(q_k)=1}} R(\epsilon,\epsilon',\{j_k,p_k,q_k\})\cdot$$

$$\cdot\prod_{k=1}^{N} < S_{t_{j_k}} f_{0,j_k}, S_{t_{p_k}} f_{\epsilon(p_k),p_k} >< S_{t_{j_k}} f_{1,j_k}, S_{t_{q_k}} f_{\epsilon'(q_k),q_k} > \tag{2.17}$$

where, R is some power of -1.

The second term in the right hand side of (2.17) consists of the terms in which there exsits k, such that the annihilation operators $A(S_{t_{j_k}} f_{0,j_k})$, $A(S_{t_{j_k}} f_{1,j_k})$ are used to produce a scalar product with the creation operators $A^+(S_{t_{p_k}} f_{\epsilon(p_k),p_k})$, $A^+(S_{t_{q_k}} f_{\epsilon(q_k),q_k})$ respectively with $p_k \ne q_k$. Such terms will be called **cross–terms**.

Since the absolute value of the second term in the right hand side of (2.17) is less than or equal to

$$\sum_{\substack{1\le j_k,p_k\ne q_k\le n,\ \ \epsilon,\epsilon'\in\{0,1\}^{2N} \\ j_k<p_k,j_k<q_k,\varepsilon(j_k)=0,\varepsilon(p_k)=\varepsilon(q_k)=1}} \prod_{k=1}^{N} | < S_{t_{j_k}} f_{0,j_k}, S_{t_{p_k}} f_{\epsilon(p_k),p_k} >< S_{t_{j_k}} f_{1,j_k}, S_{t_{q_k}} f_{\epsilon'(q_k),q_k} > | \tag{2.18}$$

Lemma (3.2) in [Lu] guarantees that this term contributes to our limit trivially, i.e. the limit of the scalar product in (3.10) can be replaced by the first term of the right hand side of (2.17).

Now, we compute the limit of the first term of the right hand side of (2.17). For each $1 \le j_k < p_k \le n$, consider the quantity

$$< \Phi, \cdots A(S_{t_{j_k}} f_{0,j_k}) A(S_{t_{j_k}} f_{1,j_k}) \cdots A(S_{t_{p_k}} f_{1,p_k}) A(S_{t_{p_k}} f_{0,p_k}) \cdots \Phi > \tag{2.19}$$

in which the annihilation operators $A(S_{t_{j_k}} f_{0,j_k})$, $A(S_{t_{j_k}} f_{1,j_k})$ are used to produce scalar products with the creation operators $A^+(S_{t_{p_k}} f_{1,p_k})$, $A^+(S_{t_{p_k}} f_{0,p_k})$. In order to do this, the annihilation operators $A(S_{t_{j_k}} f_{0,j_k})$, $A(S_{t_{j_k}} f_{1,j_k})$ must be on the left of the creation operator $A^+(S_{t_{p_k}} f_{1,p_k})$. This procedure, by the CAR, gives a factor

$$(-1)^{2(p_k-j_k-1)} \cdot (-1)^{2(p_k-j_k-1)} = 1$$

(the appearance of 2 is due to the fact that there are two operators corresponding to each time moment). By doing this for any pair $j_k < p_k$, it follows from the CAR that the first term of the right hand side of (2.10) is equal to

$$\sum_{1\le j_k<p_k\le 2N} \prod_{k=1}^{N} \lambda^2 \int_{S_{j_k}/\lambda^2}^{T_{j_k}/\lambda^2} dt_{j_k} \int_{S_{p_k}/\lambda^2}^{T_{p_k}/\lambda^2} dt_{p_k}$$

$$< \Phi, A(S_{t_{j_k}} f_{0,j_k}) A(S_{t_{j_k}} f_{1,j_k}) A(S_{t_{p_k}} f_{1,p_k}) A(S_{t_{p_k}} f_{0,p_k}) \Phi > \tag{2.20}$$

By sending λ to zero, we complete the proof using (2.8) and (2.10).

§3.) Stochastic Bosonization for interacting systems

We shall now apply the result of the previous Section to the following interaction Hamiltonian:

$$H_I := i\lambda(D \otimes A_Q^+(g_0)A_Q^+(g_1) - D^+ \otimes A_Q(g_1)A_Q(g_0)) \tag{3.1}$$

where $\lambda > 0$ is a real number, D is a bounded (but this is not really necessary) operator on a space H_S, called *the system space* and A_Q, A_Q^+ are creation and annihilation operators which we suppose realized on a representation space of the CAR–algebra over a 1–particle space $\mathcal{H}_1$. The initial state is given by

$$< \Phi_Q, \cdot\ \Phi_Q > \tag{3.2}$$

with Φ cyclic for the representation and Gaussian with 2–point function

$$< \Phi_Q, A_Q^+(f)A_Q(g)\Phi_Q >=< f, \frac{1-Q}{2} g > \tag{3.3}$$

$\Gamma_-(\mathcal{H}_1)$ where Q 1 is an operator on $\mathcal{H}_1$, for example $Q = \frac{1-ze^{-\beta H_1}}{1+ze^{-\beta H_1}}$.

It is well known that up to a unitary isomorphism

$$A_Q(f) = A(Q_+f) \otimes 1 + \theta \otimes A^+(\iota Q_- f) \tag{3.4}$$

where is the Fermi Fock space and the right hand side of (3.4) is an operator on the Hilbert space

$$\Gamma_-(\mathcal{H}) \otimes \Gamma_-(\mathcal{H}_\iota)$$

where $\mathcal{H}_\iota$ is the conjugate space of $\mathcal{H}$ (the space of *bra* vectors) and

$$Q_+ = \sqrt{\frac{1+Q}{2}}, \qquad Q_- = \sqrt{\frac{1-Q}{2}}, \qquad \theta := \bigoplus_{n=0}^{\infty}(-1)^n \tag{3.5}$$

Thus, up to isomorphism,

$$A_Q(g_1)A_Q(g_0) = A(Q_+g_1)A(Q_+g_0) \otimes 1 + A(Q_+g_1)\theta \otimes A(\iota Q_-g_0)+$$

$$+\theta A(Q_+g_0) \otimes A(\iota Q_-g_1) + 1 \otimes A(\iota Q_-g_0)A(\iota Q_-g_1) \tag{3.6}$$

By considering $2n$–point functions, one can prove that, second the third and fourth terms give trivial contribution to our WCL. So the problem is reduced to Fock case.

Thanks to the discussion in §1, we can limit ourselves to the Fock case. In this case, we have:

i) the Fermi–Fock space $\Gamma_-(\mathcal{H}_1)$,

ii) the interaction Hamiltonian

$$H_I := i\lambda(D \otimes A^+(g_0)A^+(g_1) - D^+ \otimes A(g_1)A(g_0)) \tag{3.7}$$

iii) the vacuum vector $\Phi := 1 \oplus 0 \oplus 0 \oplus \cdots$,

iv) a unitary group $\{S_t := e^{iH_1 t}\}_{t\geq 0}$ on the Hilbert space $\mathcal{H}_1$ with a self–adjoint operator H_1,

v) a subspace $\mathcal{K}$ of $\mathcal{H}$ with the property

$$\int_{-\infty}^{\infty} dt | < f, S_t g > | < \infty, \qquad \forall f, g \in \mathcal{K} \tag{3.8}$$

vi) the evolved interaction Hamiltonian

$$H_I(t) := e^{i(H_0 \otimes 1 + 1 \otimes d\Gamma(H_1))t} H_I e^{-i(H_0 \otimes 1 + 1 \otimes d\Gamma(H_1))t} \tag{3.9}$$

vii) the evolution operators $\{U_t^{(\lambda)}\}_{t\geq 0}$ defined as the solution of Schödinger equation:

$$\frac{d}{dt} U_t^{(\lambda)} = -i\lambda H_I(t) U_t^{(\lambda)} , \qquad\qquad U_0^{(\lambda)} = 1 \tag{3.10}$$

where, λ is the coupling constant.

$U_t^{(\lambda)}$ has the form

$$U_t^{(\lambda)} = 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n (-i\lambda)^n H(t_1) \cdots H(t_n) \tag{3.11}$$

which is weakly convergent on appropriate domain.

Our basic assumption is that the system Hamiltonian has discrete spectrum or, at least, that it has a complete orthonormal basis (e_n) of eigenvectors. Thus we have that

$$D(t) := e^{iH_0 t} D e^{-iH_0 t} = \sum_{n,m=0}^{\infty} e^{ix_n t} |e_n >< e_n|D|e_m >< e_m| e^{-ix_m t} =$$

$$= \sum_{n,m=0}^{\infty} e^{i(x_n - x_m)t} |e_n >< e_n|D|e_m >< e_m| \tag{3.12}$$

Denote by $\{\omega_j\}_{j=0}^{\infty}$ the set $\{x_n - x_m;\ n, m = 0, 1, 2, \cdots\}$, then $D(t)$ has the form

$$\sum_{j=0}^{\infty} e^{i\omega_j t} D_j \tag{3.13}$$

where

$$D_j := \sum_{\substack{n,m \\ x_n - x_m = \omega_j}} |e_n >< e_n|D|e_m >< e_m| \tag{3.14}$$

Notice that, with this notations we have the property: $\omega_j \neq \omega_r$ if $j \neq r$.

The limit of $U_{t/\lambda^2}^{(\lambda)}$, as $\lambda \to 0$ will be investigated in the present note.

We shall consider first of all the case of non–rotating wave approximation, i.e.

$$D(t) = e^{i\omega t} D \tag{3.15}$$

then by combining the results and techniques of the present paper with techniques now standard in the stochastic limit of quantum field theory, we shall state the result in the general case.

§4.) The limit process

In this Section we investigate the limit, as $\lambda \to 0$, of matrix elements (with respect to the collective number vectors) of the wave operator $U_t^{(\lambda)}$ i.e., expanding $U_t^{(\lambda)}$ in series:

$$< \prod_{k=1}^{N} A_\lambda^+(S_k, T_k; f_{0,k}, f_{1,k})\Phi, \sum_{n=0}^{\infty} \lambda^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$\prod_{k=1}^{n} A^{\varepsilon(k)}(S_{t_k} g_{1-\varepsilon(k)}) A^{\varepsilon(k)}(S_{t_k} g_{\varepsilon(k)}) e^{i(-1)^{1-\varepsilon(k)}\omega t_k} \prod_{k=1}^{N'} A_\lambda^+(S'_k, T'_k; f'_{0,k}, f'_{1,k})\Phi > \tag{4.1}$$

First of all, since the difference between the CCR and the CAR in a scalar product such as (4.1) consists only of some power of -1, the uniform estimate of Lemma (3.2) of [Lu] can be applied directly and we have that

THEOREM (4.1) The limit in the sum over n of the terms (4.1) can be performed term by term.

By using the notations, (2.2) and (2.7), we see the (4.1) can be rewritten as

$$\sum_{n=0}^{\infty} \lambda^{n+N+N'} \int_{S_1/\lambda^2}^{T_1/\lambda^2} du_1 \cdots \int_{S_N/\lambda^2}^{T_N/\lambda^2} du_N$$

$$\int_{S'_1/\lambda^2}^{T'_1/\lambda^2} dv_1 \cdots \int_{S'_{N'}/\lambda^2}^{T'_{N'}/\lambda^2} dv_{N'} \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$

$$e^{i\omega(\sum_{k=1}^{N'} v_k + \sum_{k=1}^{n} (-1)^{1-\varepsilon(k)} t_k - \sum_{k=1}^{N} u_k)}$$

$$< \Phi, [\prod_{k=1}^{N} \mathcal{A}^+(S_{u_k} f_{0,k}, S_{u_k} f_{1,k})]^* \prod_{k=1}^{n} \mathcal{A}^{\varepsilon(k)}(S_{t_k} g_{\varepsilon(k)}), S_{t_k} g_{1-\varepsilon(k)}) \prod_{k=1}^{N'} \mathcal{A}^+(S_{v_k} f'_{0,k}, S_{v_k} f'_{1,k})\Phi > \tag{4.7}$$

Now we shall, by applying the quasi–CCR, calculate the scalar product in (4.7). Suppose that the annihilation operator on the extreme right in the product of operators in (4.7) corresponds to the time–moment s and look at the product

$$\mathcal{A}(S_s \bar{g}_0, S_s \bar{g}_1) \mathcal{A}^+(S_\tau g'_0, S_\tau g'_1) \tag{4.8}$$

where, in the notations of (4.7):

i) if $s = u_1$, then τ is defined as t_1 and $\bar{g}_\varepsilon := f_{\varepsilon,1}$, $g'_\varepsilon := g_\varepsilon$;

ii) if $s = t_p$ for some $p \leq n-1$, then τ is defined as t_{p+1} and, in the notations of i), $\bar{g}_\varepsilon := g_\varepsilon$, $g'_\varepsilon := g_\varepsilon$;

iii) if $s = t_n$, then τ is defined as v_1 and $\bar{g}_\varepsilon := g_\varepsilon$, $g'_\varepsilon := f_{\varepsilon,1}$.

By the quasi–CCR, (4.8) is equal to

$$< (S_s\bar{g}_0, S_s\bar{g}_1), (S_\tau g'_0, S_\tau g'_1) > + \mathcal{A}^+(S_s\bar{g}_0, S_s\bar{g}_1)\mathcal{A}(S_\tau g'_0, S_\tau g'_1) +$$
$$+ R(S_s\bar{g}_0, S_s\bar{g}_1; S_\tau g'_0, S_\tau g'_1) \tag{4.9}$$

Hence the scalar product in (4.7) is equal to:

$$< \Phi, \text{a product of annihilation and creation operators}\cdot$$
$$\cdot\Big(< (S_s\bar{g}_0, S_s\bar{g}_1), (S_\tau g'_0, S_\tau g'_1) > + \mathcal{A}^+(S_s\bar{g}_0, S_s\bar{g}_1)\mathcal{A}(S_\tau g'_0, S_\tau g'_1) +$$
$$+ R(S_s\bar{g}_0, S_s\bar{g}_1; S_\tau g'_0, S_\tau g'_1)\Big) \cdot \text{a product of creation operators} \cdot \Phi > \tag{4.10}$$

Let us study the third term in (4.10), i.e.:

$$< \Phi, \text{a product of annihilation and creation operators}\cdot$$
$$\cdot R(S_s\bar{g}_0, S_s\bar{g}_1; S_\tau g'_0, S_\tau g'_1) \cdot \text{ aproductofcreationoperators } \Phi > \tag{4.11}$$

Notice that $R(S_s\bar{g}_0, S_s\bar{g}_1; S_\tau g'_0, S_\tau g'_1)$ is an operator equal to a sum of three quantities of the form $< S_s f, S_\tau f' > A^+(S_\tau g)A(S_s g')$. Therefore (4.11) is equal to a sum of three quantities like

$$< \Phi, \text{a product of annihilation and creation operators } A^+(S_\tau g)A(S_s g)$$
$$\text{a product of creation operators } \Phi >< S_s f, S_\tau f' > \tag{4.12}$$

It is certainly true that the annihilation (resp. creation) operator $A(S_s g')$ (resp. $A^+(S_\tau g)$) must be used to produce a scalar product with a creation operator $A^+(S_u h')$ (resp. $A^+(S_v h)$) with $u \neq s$ (resp. $v \neq \tau$). In other words, (4.12) is equal to a sum of cross–terms and therefore contributes to our limit only zero. That is, up to an $o(1)$, (4.10) is equal to

$$< \Phi, \text{a product of annihilation and creation operators } \cdot$$
$$\Big(< (S_s\bar{g}_0, S_s\bar{g}_1), (S_\tau g'_0, S_\tau g'_1) > + \mathcal{A}^+(S_s\bar{g}_0, S_s\bar{g}_1)\mathcal{A}(S_\tau g'_0, S_\tau g'_1)\Big)\cdot$$
$$\text{a product of creation operators } \Phi > \tag{4.13}$$

Repeating the above discussion, we find that although $\mathcal{A}^\pm$ are not Boson objects, as $\lambda \to 0$, the remainder R can be effectively forgotten and they satisfy the CCR. Therefore,

the following result can be obtained by combining the above discussions and the techniques in [Lu]:

THEOREM (4.2) For any $\xi, \eta \in \mathcal{H}_0$, the limit, as $\lambda \to 0$, of

$$< \xi \otimes \prod_{k=1}^{N} A_\lambda^+(S_k, T_k; f_{0,k}, f_{1,k})\Phi, U_{t/\lambda^2}^{(\lambda)}\eta \otimes \prod_{k=1}^{N'} A_\lambda^+(S'_k, T'_k; f'_{0,k}, f'_{1,k})\Phi > \tag{4.13}$$

exsits and is equal to

$$< \xi \otimes \prod_{k=1}^{N} a^+(\chi_{[S_k,T_k]} \otimes f_{0,k} \otimes_{\mathrm{FB}} f_{1,k})\Psi, U(t) \prod_{k=1}^{N'} a^+(\chi_{[S'_k,T'_k]} \otimes f'_{0,k} \otimes_{\mathrm{FB}} f'_{1,k})\Psi > \tag{4.14}$$

where, $U(t)$ is the solution of the Boson stochastic differential equation

$$U(t) = 1 + \int_0^t \Big(D \otimes da_s^+(g_0 \otimes_{\mathrm{FB}} g_1) - D^+ \otimes da_s(g_0 \otimes_{\mathrm{FB}} g_1) -$$

$$-D^+ D \otimes 1 (g_0 \otimes_{\mathrm{FB}} g_1 | g_0 \otimes_{\mathrm{FB}} g_1)_- \Big) U(s) \tag{4.15}$$

and where the half scalar product is defined as

$$(f \otimes_{\mathrm{FB}} g | f' \otimes_{\mathrm{FB}} g')_- := \int_{-\infty}^0 dt e^{i\omega t} \big(< f, S_t f' >< g, S_t g' > - < f, S_t g' >< g, S_t f' > \big) \tag{4.16}$$

Moreover, $\{U(t)\}_{t \geq 0}$ is a unitary process.

§5.) Discrete spectrum case

Now we discuss the problem in the more general case in which one replaces the assumption (3.9) by (3.6). In this case, the collective creation operators are defined as:

$$A_\lambda^+(S,T;\{f_0^{(j)},f_1^{(j)}\}) := \sum_{j\in\mathbf{Z}} \lambda \int_{S/\lambda^2}^{T/\lambda^2} e^{i\omega_j t} A^+(S_t f_0^{(j)}) A^+(S_t f_1^{(j)}) dt \tag{5.1}$$

for $S \le T$ and $f_0^{(j)}, f_1^{(j)} \in \mathcal{K}$ with the property that there are only a finite number of j such that $f_0^{(j)} \ne 0, f_1^{(j)} \ne 0$. The analogue of Theorem (3.2) will then be

THEOREM (5.1) The limit, as $\lambda \to 0$, of

$$< \Phi, \prod_{k=1}^n A_\lambda^{\varepsilon(k)}(S_k,T_k;f_{0,k}^{(j)},f_{1,k}^{(j)})\Phi > \tag{5.2}$$

exsits and equal to

$$< \Psi, \prod_{k=1}^n a^{\varepsilon(k)}(\chi_{[S_k,T_k]} \otimes \bigoplus_{j\in\mathbf{Z}} f_{0,k}^{(j)} \otimes_{\mathrm{FB}} f_{1,k}^{(j)})\Psi > \tag{5.3}$$

where, a, a^+ and Ψ are (the Boson) annihilation, creation operators on the Boson–Fock space $\Gamma_+(L^2(\mathbf{R}) \otimes \bigoplus_{j\in\mathbf{Z}} \mathcal{K} \otimes_{\mathrm{FB}}^{(j)} \mathcal{K})$ respectively and Ψ is the vacuum vector and for each $j \in \mathbf{Z}$

$$(f_0 \otimes_{\mathrm{FB}}^{(j)} f_1 | f_0' \otimes_{\mathrm{FB}}^{(j)} f_1') :=$$

$$= \int_{-\infty}^{\infty} ds e^{i\omega_j s} [< f_0, S_s f_1' >< f_1, S_s f_0' > - < f_0, S_s f_0' >< f_1, S_s f_1' >] \tag{5.4}$$

The final result is

THEOREM (5.2) For any $\xi, \eta \in \mathcal{H}_0$, the limit, as $\lambda \to 0$, of

$$< \xi \otimes \prod_{k=1}^N A_\lambda^+(S_k,T_k;\{f_{0,k}^{(j)},f_{1,k}^{(j)}\})\Phi, U_{t/\lambda^2}^{(\lambda)}\eta \otimes \prod_{k=1}^{N'} A_\lambda^+(S_k',T_k';\{h_{0,k}^{(j)},h_{1,k}^{(j)})\Phi > \tag{5.5}$$

exsits and is equal to

$$< \xi \otimes \prod_{k=1}^N a^+(\chi_{[S_k,T_k]} \otimes \bigoplus_{j\in\mathbf{Z}} f_{0,k}^{(j)} \otimes_{\mathrm{FB}}^{(j)} f_{1,k}^{(j)})\Psi,$$

$$U(t)\prod_{k=1}^{N'} a^+(\chi_{[S'_k,T'_k]}\otimes\bigoplus_{j\in\mathbf{Z}} h_{0,k}^{(j)}\otimes_{\mathrm{FB}}^{(j)} h_{1,k}^{(j)})\Psi > \tag{5.6}$$

where, $U(t)$ is the solution of the Boson stochastic differential equation

$$U(t)=1+\int_0^t\sum_{j\in\mathbf{Z}}\Big(D_j\otimes da_s^+(0\bigoplus g_0\otimes_{\mathrm{FB}}^{(j)} g_1\bigoplus 0)-D_j^+\otimes da_s(0\bigoplus g_0\otimes_{\mathrm{FB}}^{(j)} g_1\bigoplus 0)-$$

$$-D_j^+D_j\otimes 1(g_0\otimes_{\mathrm{FB}}^{(j)} g_1|g_0\otimes_{\mathrm{FB}}^{(j)} g_1)_-\Big)U(s) \tag{5.7}$$

and the half scalar product is defined as

$$(f\otimes_{\mathrm{FB}}^{(j)} g|f'\otimes_{\mathrm{FB}}^{(j)} g')_- := \int_{-\infty}^0 dte^{i\omega_j t}\big(< f,S_tf' >< g,S_tg' > - < f,S_tg' >< g,S_tf' >\big) \tag{5.8}$$

Moreover, $\{U(t)\}_{t\geq 0}$ is a unitary process.

Proof The theorem can proved by combining the results in the present note with those in [Lu].

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